❖ Quantum Mechanical Operators and Their Commutation Relations

An operator may be simply defined as a mathematical procedure or instruction which is carried out over a function to yield another function.

$$\text{(Operator) . (Function) = (Another function)} \quad (67)$$

The function used on the left-hand side of the equation (67) is called as the operand i.e. the function over which the operation is actually carried out. The operator alone has no significance but when operated over a certain mathematical description, these operators can provide very detailed insights into those functions. Some of the simple illustrations of equation (67) are given below.

i) Consider the differential operator d/dx whose operation has to be studied over the function $y = x^5$. The mathematical treatment is

$$\frac{dy}{dx} = \frac{d}{dx}x^5 = 5x^4 \quad (68)$$

The operation of d/dx on y means that the rate of change of function y w.r.t. the variable x. The expression x^5 is the operand while the $5x^4$ is the final result of our differential operator.

ii) Consider the integral operator $\int (y)\, dx$ whose operation has to be studied over the function $y = x^5$. The mathematical treatment is

$$\int y(dx) = \int x^5(dx) = \frac{x^6}{6} \quad (69)$$

The operation of $\int dx$ on y means that we can find the function whose derivative is x^5. The expression x^5 is the operand while the $x^6/6$ is the final result of our integral operator.

In a similar way, the multiplication of a function by a constant number, or taking the square and cube roots of any function are also the operators which give some other function after operating them over the operand. The symbol of the operator typically carries a cap over it (Â) which differentiates it from the function used in the whole procedure.

➢ ***Algebra of Operators***

Just like the normal algebra, the resultants like addition or the multiplication of operators also follow certain rules; however, these rules are different from the typical algebra. Some of the most important rules of operator algebra are given below.

1. Addition and subtraction of operators: Let A and B as two different operators; *f* as the function that has to be used as the operand. Then, the addition and subtraction of these two operators must be carried out in the manner discussed below.

$$(\hat{A} + \hat{B})f = \hat{A}f + \hat{B}f \tag{70}$$

and

$$(\hat{A} - \hat{B})f = \hat{A}f - \hat{B}f \tag{71}$$

2. Multiplication of operators: If A and B as two different operators; and *f* as the function that has to be used as operand. Then, the multiplication of these two operators must be carried out in the manner discussed below.

$$\hat{A}\hat{B}f = f'' \tag{72}$$

The interpretation of the above equation is that first we need to operate B on *f*, which would give us another function *f*′, which in turn is further used as the operand for operator giving the final result *f*″. In other words, we can say that when multiplication of two or more operators is used, we should follow from left to right. Moreover, the square or cube of a particular operator must be considered as double or triple multiplication of the operator itself; mathematically, it can be shown as given below.

$$\hat{A}^2 f = \hat{A}\hat{A}f \tag{73}$$

At this point it also very important to discuss one of the most fundamental properties of operator multiplication, the commutation relation or the commutation rule. Consider two operators, A and B which can be operated over the function *f*.

$$\hat{A} = \frac{d}{dx}; \quad \hat{B} = x; \quad f = x^3 \tag{74}$$

Now

$$\hat{A}\hat{B}f = \frac{d}{dx}x(x^3) = \frac{d}{dx}x^4 = 4x^3 \tag{75}$$

And

$$\hat{B}\hat{A}f = x\frac{d}{dx}(x^3) = x(3x^2) = 3x^3 \tag{76}$$

From equation (75) and (76), it the clear that in this case

$$\hat{A}\hat{B}f \neq \hat{B}\hat{A}f \tag{77}$$

These operators are said to be non-commutating with the commutator given below.

$$\hat{A}\hat{B} - \hat{B}\hat{A} = 4x^3 - 3x^3 \tag{78}$$

However, the two operators are said to be commute if their result is the same even after reverting their order of application. Mathematically, it can be stated as given by equation (79).

$$\hat{A}\hat{B}f = \hat{B}\hat{A}f \tag{79}$$

This is quite different from the normal algebra in which the product of two numbers is always the same irrespective of the order of multiplication ($x.y = y.x$). Summarizing the commutation rule, it can be concluded that

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0 \ \rightarrow Commutating \tag{80}$$

and

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0 \ \rightarrow Non\text{-}commutating \tag{81}$$

3. Linear Operator: An operator Â is said to be a linear operator if its application on the sum of two functions f and g gives the same result as the sum of its individual operations. Mathematically, it can be shown as given below.

$$\hat{A}(f + g) = \hat{A}f + \hat{A}g \tag{82}$$

For example, consider the differential operator A; with f and g as the functions which have to be used as the operand.

$$\hat{A} = \frac{d}{dx}; \quad f = 2x^2; \quad g = 3x^2 \tag{83}$$

or

$$\hat{A}(f + g) = \frac{d}{dx}(2x^2 + 3x^2) = \frac{d}{dx}(5x^2) = 10x \tag{84}$$

or

$$\hat{A}f + \hat{A}g = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x^2) = 4x + 6x = 10x \tag{85}$$

Hence, from equation (84) and equation (85), it is clear that the differential operator is clearly linear in nature. On the other hand, the "square root" operator is not linear as it does not give the same result when operated individually.

➢ *Some Important Quantum Mechanical Operators*

One of the most basic and very popular operators in quantum mechanics is the Laplacian operator, typically symbolized as ∇^2, and is given by the following expression.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{86}$$

The popular form of the Schrodinger equation can be written in terms of Laplacian operator as well.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \tag{87}$$

or

$$\nabla^2 \psi + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \tag{88}$$

The Laplacian operator is pronounced as "del squared". This operator is also a part of the "mighty" Hamiltonian operator which forms the basis for value evaluation for other operators, as we have already discussed in the postulates of quantum mechanics. The Hamiltonian operator is typically symbolized as $\hat{H}$ and is given by the following expression.

$$\hat{H} = -\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V \tag{89}$$

or

$$\hat{H} = -\frac{h^2}{8\pi^2 m}\nabla^2 + V \tag{90}$$

The popular form of the Schrodinger equation is written in terms of the Hamiltonian operator as well.

$$\hat{H}\psi = E\psi \tag{91}$$

or

$$\left[-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V\right]\psi = E\psi \tag{92}$$

or

$$\left(-\frac{h^2}{8\pi^2 m}\nabla^2 + V\right)\psi = E\psi \tag{93}$$

Furthermore, we know from the third postulate of quantum mechanics that owing to the constant value of E (eigenvalue) the wave function ψ can be labeled as eigenfunction.

Therefore, the Schrodinger equation is also called as the "eigen value equation". Simplifying this, we can say that

$$(Energy\ operator)(Wave\ function) = (Energy)(Wave\ function) \tag{94}$$

The equation (94) is applicable to observables in the quantum mechanical world.

For three dimensional systems, like the Hamiltonian, the operator can be obtained by summing the individual operators along three different axes. For instance, some important three-dimensional operators are:

$$\hat{T} = \frac{-h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{95}$$

$$\hat{p} = \frac{h}{2\pi i}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \tag{96}$$

The list of various important quantum mechanical operators in one dimension, along with their mode of operation is given below.

Table 2. Name and symbols of various important physical properties and their corresponding quantum mechanical operators.

Physical property		Operator	
Name	*Symbol*	*Symbol*	*Operation*
Position	x	$\hat{x}$	Multiplication by x
Position squared	x^2	$\hat{x}^2$	Multiplication by x^2
Position cubed	x^3	$\hat{x}^2$	Multiplication by x^3
Momentum	p_x	$\hat{p}_x$	$\frac{h}{2\pi i}\frac{\partial}{\partial x}$
Momentum squared	p_x^2	$\hat{p}_x^2$	$\frac{-h^2}{4\pi^2}\frac{\partial^2}{\partial x^2}$
Kinetic energy	$T = \frac{P^2}{2m}$	$\hat{T}_x$	$\frac{-h^2}{8\pi^2 m}\frac{\partial^2}{\partial x^2}$
Potential energy	$V(x)$	$\hat{V}(x)$	Multiplication by $V(x)$
Total energy	$E = T + V(x)$	$\hat{H}$	$\frac{-h^2}{8\pi^2 m}\frac{\partial^2}{\partial x^2} + V(x)$

Besides the record of different operators presented in 'Table 2', there still many operators which are extremely important like angular momentum, parity, or the step-up–step-down operators. The discussion of every operator is beyond the scope of this book; however, a brief discussion of the essential operators in quantum mechanics is given below.

1. Angular momentum operator: In order to understand the angular momentum operator in the quantum mechanical world, we first need to understand the classical mechanics of one particle angular momentum. Let us consider a particle of mass *m* which moves within a cartesian coordinate system with a position vector "*r*". Hence, we can say that

$$r = ix + jy + kz \tag{97}$$

The coordinates *x*. *y* and *z* are the functions of time, and therefore, we can define the velocity as the time derivative of the position vector as given below.

$$v = \frac{dr}{dt} = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt} \tag{98}$$

or

$$v = v_x + v_y + v_z \tag{99}$$

Now, since we that $p = mv$, we can say that

$$p_x = mv_x; \quad p_y = mv_y; \quad p_z = mv_z \tag{100}$$

The angular momentum of a particle with mass *m* and distance *r* from the origin is given by the following relation.

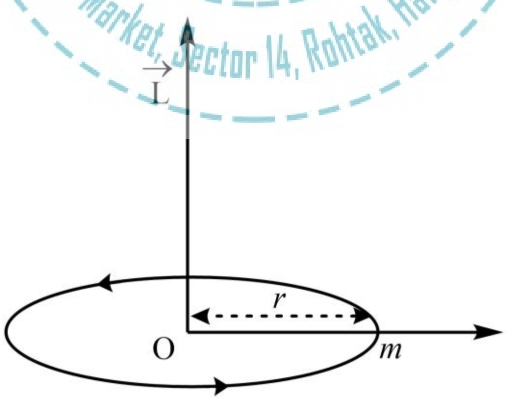

Figure 6. The angular momentum vector.

$$\vec{L} = \vec{v} \times m \times \vec{r} \tag{101}$$

$$\vec{L} = \vec{p} \times \vec{r} \tag{102}$$

Equation (102) can also be written in the form of a matrix as:

$$L = \begin{bmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{bmatrix} \tag{103}$$

$$L_x = yp_z - zp_y; \quad L_y = zp_x - xp_z; \quad L_z = xp_y - yp_x \tag{104}$$

Where i, j, k are the unit vectors along x, y, z axis and L_x, L_y, L_z are the component of angular momentum along x, y, z axis. Moreover, it is also worthy to note that the angular momentum vector is always perpendicular to the direction of the position vector of the particle i.e. the plane in which the particle is moving.

Now since the mathematical nature of any quantum mechanical operator is dependent upon the classical expression of the same observable, the angular momentum is not any exception. The quantum mechanical operator for angular momentum is given below.

$$\hat{L} = -i\frac{h}{2\pi}(r \times \nabla) = -i\hbar(r \times \nabla) \tag{105}$$

The angular momentum can be divided into two categories; one is orbital angular momentum (due to the orbital motion of the particle) and the other is spin angular momentum (due to spin motion of the particle). Moreover, being a vector quantity, the operator of angular momentum can also be resolved along different axes.

$$\hat{L} = \hat{L}_x + \hat{L}_y + \hat{L}_z \tag{106}$$

And we know that

$$\hat{L}_x = yp_z - zp_y = y\left(\frac{h}{2\pi i}\frac{\partial}{\partial z}\right) - z\left(\frac{h}{2\pi i}\frac{\partial}{\partial y}\right) = \frac{h}{2\pi i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{107}$$

or

$$\hat{L}_y = zp_x - xp_z = z\left(\frac{h}{2\pi i}\frac{\partial}{\partial x}\right) - x\left(\frac{h}{2\pi i}\frac{\partial}{\partial z}\right) = \frac{h}{2\pi i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{108}$$

or

$$\hat{L}_z = xp_y - yp_x = x\left(\frac{h}{2\pi i}\frac{\partial}{\partial y}\right) - y\left(\frac{h}{2\pi i}\frac{\partial}{\partial x}\right) = \frac{h}{2\pi i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{109}$$

$$\hat{L} = \frac{h}{2\pi i}\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \tag{110}$$

It is also worthy to recall that equation (107) to (110) can also be reported in terms of ħ; or by multiplying and dividing by i, or both.

2. Ladder operator: These operators are also called as step-up–step-down or rising-lowering operators. The reason for such terminology lies in the fact that these operators can increase or decrease the eigenvalues. Moreover, it should also be noted that this increase or decrease is always quantized in nature.

$$\widehat{J_+} = \widehat{J_x} + i\widehat{J_y} \tag{111}$$

and

$$\widehat{J_+} = \widehat{J_x} - i\widehat{J_y} \tag{112}$$

The equation (111) and (112) represent the step-up and step-down operators respectively. These operators can be used to increase or decrease the eigen values.

➢ ***Operator Evaluation***

The operator evaluation simply means that we need to find the result by applying the operator over a given function. Some general examples are given below.

i) (d/dx) (x^5): In this case d/dx is the operator while the function x^5 is the operand.

$$\frac{d}{dx}x^5 = 5x^4 \tag{113}$$

ii) $\int(x^5)$: In this case, $\int$ is the operator while the function x^5 is the operand.

$$\int x^5 = \frac{x^6}{6} \tag{114}$$

iii) (d^2/dt^2) (A*Sine* 2π*vt*): In this particular case, (d^2/dt^2) is the operator while the function (A *Sin* 2π*vt*) is the operand.

Let the function is symbolized by *y*. Then, we have

$$y = A\,Sin\,2\pi vt \tag{115}$$

Differentiating with respect to *t*, we get

$$\frac{dy}{dt} = A\,2\pi v\,Cos\,2\pi vt \tag{116}$$

Differentiating again

$$\frac{d^2y}{dt^2} = -A\,4\pi^2v^2\,Sin\,2\pi vt \tag{117}$$

The operator evaluation is frequently used as a part of the commutator calculation and will be discussed in detail in this chapter.

➢ *Calculation of Resultant Operator*

Sometimes the operator is simplified to another form which is easy to apply over a function. This resultant operator is obtained by the rules of operator algebra. For instance, consider the following cases.

i) Find the resultant expression for the following operator

$$\left(\frac{d}{dx}x\right)^2 \tag{118}$$

In order to find the resultant operator, suppose a function ψ(x) which is used as an operand, then we can say

$$\left(\frac{d}{dx}x\right)^2 \psi = \left(\frac{d}{dx}x\right)\left(\frac{d}{dx}x\right)\psi \tag{119}$$

or

$$\left(\frac{d}{dx}x\right)^2 \psi = \left(\frac{d}{dx}x\right)\left(\frac{d}{dx}x\psi\right) \tag{120}$$

or

$$\left(\frac{d}{dx}x\right)^2 \psi = \left(\frac{d}{dx}x\right)\left(x\frac{d\psi}{dx} + \psi\frac{dx}{dx}\right) \tag{121}$$

$$\left(\frac{d}{dx}x\right)^2 \psi = \frac{d}{dx}\left(x^2\frac{d\psi}{dx} + x\psi\right) \tag{122}$$

$$\left(\frac{d}{dx}x\right)^2 \psi = \left[x^2\frac{d^2\psi}{dx^2} + \frac{d\psi}{dx}(2x)\right] + \left[x\frac{d\psi}{dx} + \psi\frac{dx}{dx}\right] \tag{123}$$

$$\left(\frac{d}{dx}x\right)^2 \psi = x^2\frac{d^2\psi}{dx^2} + 2x\frac{d\psi}{dx} + x\frac{d\psi}{dx} + \psi \tag{124}$$

$$\left(\frac{d}{dx}x\right)^2 \psi = \left[x^2\frac{d^2}{dx^2} + 3x\frac{d}{dx} + 1\right]\psi \tag{125}$$

Removing ψ from both sides, we get

$$\left(\frac{d}{dx}x\right)^2 = x^2\frac{d^2}{dx^2} + 3x\frac{d}{dx} + 1 \tag{126}$$

ii) Find the resultant expression for the following operator

$$\left(x + \frac{d}{dx}\right)\frac{d}{dx} \tag{127}$$

In order to find the resultant operator, suppose a function ψ(x) which is used as operand, then we can say that

$$\left[\left(x+\frac{d}{dx}\right)\frac{d}{dx}\right]\psi=\left(x+\frac{d}{dx}\right)\frac{d\psi}{dx} \tag{128}$$

$$\left[\left(x+\frac{d}{dx}\right)\frac{d}{dx}\right]\psi=x\frac{d\psi}{dx}+\frac{d^2\psi}{dx^2}$$

Removing ψ from both sides, we get

$$\left(x+\frac{d}{dx}\right)\frac{d}{dx}=x\frac{d}{dx}+\frac{d^2}{dx^2} \tag{129}$$

iii) Find the resultant expression for the following operator

$$\left(\frac{d}{dx}+x\right)^2 \tag{130}$$

In order to find the resultant operator, suppose a function ψ(*x*) which is used as operand, then we can say that

$$\left[\left(\frac{d}{dx}+x\right)^2\right]\psi=\left[\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)\right]\psi \tag{131}$$

$$\left[\left(\frac{d}{dx}+x\right)^2\right]\psi=\left(\frac{d}{dx}+x\right)\left(\frac{d\psi}{dx}+x\psi\right) \tag{132}$$

$$\left[\left(\frac{d}{dx}+x\right)^2\right]\psi=\frac{d^2\psi}{dx^2}+\frac{d}{dx}x\psi+x\frac{d\psi}{dx}+x^2\psi \tag{133}$$

$$\left[\left(\frac{d}{dx}+x\right)^2\right]\psi=\frac{d^2\psi}{dx^2}+x\frac{d\psi}{dx}+\psi\frac{dx}{dx}+x\frac{d\psi}{dx}+x^2\psi \tag{134}$$

$$\left[\left(\frac{d}{dx}+x\right)^2\right]\psi=\frac{d^2\psi}{dx^2}+2x\frac{d\psi}{dx}+x^2\psi+\psi \tag{135}$$

Removing ψ from both sides, we get

$$\left(\frac{d}{dx}+x\right)^2=\frac{d^2}{dx^2}+2x\frac{d}{dx}+x^2+1 \tag{136}$$

iv) Find the resultant expression for the following operator

$$\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right) \tag{137}$$

In order to find the resultant operator, suppose a function ψ(*x*) which is used as operand, then we can say that

$$\left[\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)\right]\psi=\left(x+\frac{d}{dx}\right)\left(x\psi-\frac{d\psi}{dx}\right) \tag{138}$$

$$\left[\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)\right]\psi=xx\psi-x\frac{d\psi}{dx}+\frac{d}{dx}x\psi-\frac{d^2\psi}{dx^2} \tag{139}$$

$$\left[\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)\right]\psi=x^2\psi-x\frac{d\psi}{dx}+x\frac{d\psi}{dx}+\psi\frac{dx}{dx}-\frac{d^2\psi}{dx^2} \tag{140}$$

$$\left[\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)\right]\psi=x^2\psi+\psi\frac{dx}{dx}-\frac{d^2\psi}{dx^2} \tag{141}$$

$$\left[\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)\right]\psi=\left[x^2+\frac{dx}{dx}-\frac{d^2}{dx^2}\right]\psi \tag{142}$$

Removing ψ from both sides, we get

$$\left(x+\frac{d}{dx}\right)\left(x-\frac{d}{dx}\right)=x^2+1-\frac{d^2}{dx^2} \tag{143}$$

The resultant operator calculation is frequently used as a part of the commutator calculation and will be discussed in detail in this chapter.

➢ ***Commutation Relations of Various Quantum Mechanical Operators***

As we have discussed previously that one of the most fundamental properties of operator multiplication is the commutation relation or the commutation rule. two operators, A and B, are said to be commutating or non-commutating depending upon the value of their commutator.

$$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}=0 \ \rightarrow \textit{Commutating} \tag{144}$$

$$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}\neq 0 \ \rightarrow \textit{Non-commutating} \tag{145}$$

The physical significance of the commutation relations is that when two operators commute, it means they are having a simultaneous set of eigenfunctions; and their corresponding physical properties can be calculated simultaneously and accurately. However, if the commutator is non-zero, the respective physical properties cannot be obtained simultaneously and accurately. Some important commutation relations are given below.

1. Commutators of some simple operators:

i) Calculate the commutator of the following

$$\left[x,\frac{d}{dx}\right] \tag{146}$$

Let it be operated over a function ψ. We have

$$\left[x,\frac{d}{dx}\right]\psi = x\,\frac{d}{dx}\psi - \frac{d}{dx}x\,\psi \tag{147}$$

$$\left[x,\frac{d}{dx}\right]\psi = x\,\frac{d\psi}{dx} - \psi - x\frac{d\,\psi}{dx} \tag{148}$$

$$\left[x,\frac{d}{dx}\right]\psi = -\psi \tag{149}$$

or

$$\left[x,\frac{d}{dx}\right] = -1 \tag{150}$$

ii) Calculate the commutator of the following

$$\left[y,\frac{d}{dx}\right] \tag{151}$$

Let it be operated over a function ψ. We have

$$\left[y,\frac{d}{dx}\right]\psi = y\,\frac{d}{dx}\psi - \frac{d}{dx}y\,\psi \tag{152}$$

$$\left[y,\frac{d}{dx}\right]\psi = y\,\frac{d\psi}{dx} - y\frac{d\,\psi}{dx} - \psi\frac{d\,y}{dx} \tag{153}$$

$$\left[x,\frac{d}{dx}\right]\psi = 0 \tag{154}$$

iii) Calculate the commutator of the following

$$\left[\frac{d}{dx},\frac{d^2}{dx^2}\right] \tag{155}$$

Let it be operated over a function ψ. We have

$$\left[\frac{d}{dx},\frac{d^2}{dx^2}\right]\psi = \frac{d}{dx}\frac{d^2}{dx^2}\psi - \frac{d^2}{dx^2}\frac{d}{dx}\,\psi \tag{156}$$

or

$$\left[\frac{d}{dx},\frac{d^2}{dx^2}\right]\psi = \frac{d^3\psi}{dx^3} - \frac{d^3\psi}{dx^3} \tag{157}$$

$$\left[\frac{d}{dx},\frac{d^2}{dx^2}\right]\psi = 0 \tag{158}$$

2. Commutators of position and linear momentum operators:

i) Find the commutator of the following

$$[\hat{x}, \hat{p}_x] \tag{159}$$

Let it be operated over a function ψ. We have

$$[\hat{x}, \hat{p}_x]\psi = \hat{x}\,\hat{p}_x\,\psi - \hat{p}_x\,\hat{x}\;\psi \tag{160}$$

$$[\hat{x}, \hat{p}_x\,]\psi = x\frac{h}{2\pi i}\frac{\partial}{\partial x}\psi - \frac{h}{2\pi i}\frac{\partial}{\partial x}x\,\psi \tag{161}$$

$$[\hat{x}, \hat{p}_x\,]\psi = \frac{h}{2\pi i}x\frac{\partial\psi}{\partial x} - \frac{h}{2\pi i}x\frac{\partial\psi}{\partial x} - \frac{h}{2\pi i}\psi\frac{\partial x}{\partial x} \tag{162}$$

$$[\hat{x}, \hat{p}_x\,]\psi = -\frac{h}{2\pi i}\psi$$

$$[\hat{x}, \hat{p}_x] = -\frac{h}{2\pi i} = \frac{hi}{2\pi} = i\hbar \tag{163}$$

ii) Find the commutator of the following

$$[\hat{x}^n, \hat{p}_x] \tag{164}$$

Let it be operated over a function ψ. We have

$$[\hat{x}^n, \hat{p}_x\,]\psi = \hat{x}^n\,\hat{p}_x\,\psi - \hat{p}_x\,\hat{x}^n\;\psi \tag{165}$$

$$[\hat{x}^n, \hat{p}_x\,]\psi = x^n\frac{h}{2\pi i}\frac{\partial}{\partial x}\psi - \frac{h}{2\pi i}\frac{\partial}{\partial x}x^n\,\psi \tag{166}$$

$$[\hat{x}^n, \hat{p}_x\,]\psi = \frac{h}{2\pi i}x^n\frac{\partial\psi}{\partial x} - \frac{h}{2\pi i}x^n\frac{\partial\psi}{\partial x} - \frac{h}{2\pi i}nx^{n-1}\psi \tag{167}$$

$$[\hat{x}^n, \hat{p}_x\,]\psi = -\frac{h}{2\pi i}nx^{n-1}\psi \tag{168}$$

Removing ψ from both sides, we get

$$[\hat{x}^n, \hat{p}_x\,] = -\frac{h}{2\pi i}nx^{n-1} \tag{169}$$

The commutation relations between position and linear momentum can mainly be divided into three categories as discussed below.

(a) When position and momentum are along the same axis:

$$[\hat{x}^n, \hat{p}_x] = ni\hbar x^{n-1} \tag{170}$$

$$[\hat{p}_x, \hat{x}^n] = -ni\hbar x^{n-1} \tag{171}$$

and

$$[\hat{x}, \hat{p}_x^n] = ni\hbar p_x^{n-1} \tag{172}$$

$$[\hat{p}_x^n, \hat{x}] = -ni\hbar p_x^{n-1} \tag{173}$$

(b) When position and momentum are along different axis:

$$[\hat{x}, \hat{p}_y] = 0 \tag{174}$$

$$[\hat{x}, \hat{p}_z] = 0 \tag{175}$$

$$[\hat{y}, \hat{p}_x] = 0 \tag{176}$$

$$[\hat{y}, \hat{p}_z] = 0 \tag{177}$$

$$[\hat{z}, \hat{p}_x] = 0 \tag{178}$$

$$[\hat{z}, \hat{p}_y] = 0 \tag{179}$$

(b) When positions are along the different axis:

$$[\hat{x}, \hat{y}] = 0 \tag{180}$$

$$[\hat{x}, \hat{z}] = 0 \tag{181}$$

$$[\hat{y}, \hat{z}] = 0 \tag{182}$$

(b) When positions are along the different axis:

$$[\hat{p}_x, \hat{p}_y] = 0 \tag{183}$$

$$[\hat{p}_x, \hat{p}_z] = 0 \tag{184}$$

$$[\hat{p}_y, \hat{p}_z] = 0 \tag{185}$$

3. Commutators of angular momentum operators:

i) The commutator of orbital angular momentum operators along *x* and *y*-axis.

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x \tag{186}$$

Finding the values of $\hat{L}_x\hat{L}_y$, we get

$$\hat{L}_x\hat{L}_y = \left[\frac{h}{2\pi i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\left[\frac{h}{2\pi i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] \tag{187}$$

$$= -\frac{h^2}{4\pi^2}\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] \tag{188}$$

$$= -\frac{h^2}{4\pi^2}\left(y\frac{\partial}{\partial z}z\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}x\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}x\frac{\partial}{\partial z}\right) \tag{189}$$

$$= -\frac{h^2}{4\pi^2}\left(y\frac{\partial}{\partial x}\frac{\partial z}{\partial z} + yz\frac{\partial^2}{\partial^2 zx} - z^2\frac{\partial^2}{\partial^2 yx} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial yz}\right) \tag{190}$$

$$= -\hbar^2\left(y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial^2 zx} - z^2\frac{\partial^2}{\partial^2 yx} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial yz}\right) \tag{191}$$

Similarly obtaining the value of $\hat{L}_y\hat{L}_x$, we get

$$\hat{L}_y\hat{L}_x = \left[\frac{h}{2\pi i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\left[\frac{h}{2\pi i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \tag{192}$$

$$= -\frac{h^2}{4\pi^2}\left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \tag{193}$$

$$= -\frac{h^2}{4\pi^2}\left(z\frac{\partial}{\partial x}y\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}z\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}y\frac{\partial}{\partial z} + x\frac{\partial}{\partial z}z\frac{\partial}{\partial y}\right) \tag{194}$$

$$= -\frac{h^2}{4\pi^2}\left(zy\frac{\partial^2}{\partial^2 xz} - z^2\frac{\partial^2}{\partial^2 xy} - xy\frac{\partial^2}{\partial z^2} + xz\frac{\partial^2}{\partial zy} + x\frac{\partial}{\partial y}\frac{\partial z}{\partial z}\right) \tag{195}$$

$$= -\hbar^2\left(zy\frac{\partial^2}{\partial^2 xz} - z^2\frac{\partial^2}{\partial^2 xy} - xy\frac{\partial^2}{\partial z^2} + xz\frac{\partial^2}{\partial zy} + x\frac{\partial}{\partial y}\right) \tag{196}$$

Now putting the values of $\hat{L}_x\hat{L}_y$ and $\hat{L}_y\hat{L}_x$ in equation (183), we get the following.

$$[\hat{L}_x, \hat{L}_y] = \left[-\hbar^2\left(y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial^2 zx} - z^2\frac{\partial^2}{\partial^2 yx} - yx\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial yz}\right)\right] - \left[-\hbar^2\left(zy\frac{\partial^2}{\partial^2 xz} - z^2\frac{\partial^2}{\partial^2 xy} - xy\frac{\partial^2}{\partial z^2} + xz\frac{\partial^2}{\partial zy} + x\frac{\partial}{\partial y}\right)\right] \tag{197}$$

$$[\hat{L}_x, \hat{L}_y] = -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \tag{198}$$

Taking negative sign common, we get

$$[\hat{L}_x, \hat{L}_y] = \hbar^2\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{199}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\left[-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \tag{200}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \tag{201}$$

ii) The commutator of orbital angular momentum operators along *y* and *z*-axis.

$$[\hat{L}_y, \hat{L}_z] = \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y \tag{202}$$

Finding the values of $\hat{L}_y\hat{L}_z$, we get

$$\hat{L}_y\hat{L}_z = \left[\frac{h}{2\pi i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\left[\frac{h}{2\pi i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \tag{203}$$

$$= -\frac{h^2}{4\pi^2}\left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \tag{204}$$

$$= -\frac{h^2}{4\pi^2}\left(z\frac{\partial}{\partial x}x\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}x\frac{\partial}{\partial y} - z\frac{\partial}{\partial x}y\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}y\frac{\partial}{\partial x}\right) \tag{205}$$

$$= -\frac{h^2}{4\pi^2}\left(z\frac{\partial}{\partial y}\frac{\partial x}{\partial x} + zx\frac{\partial^2}{\partial xy} - x^2\frac{\partial^2}{\partial^2 zy} - zy\frac{\partial}{\partial x^2} + xy\frac{\partial^2}{\partial zx}\right) \tag{206}$$

$$= -\hbar^2\left(z\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial xy} - x^2\frac{\partial^2}{\partial^2 zy} - zy\frac{\partial}{\partial x^2} + xy\frac{\partial^2}{\partial zx}\right) \tag{207}$$

Similarly obtaining the value of $\hat{L}_z\hat{L}_y$, we get

$$\hat{L}_z\hat{L}_y = \left[\frac{h}{2\pi i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\left[\frac{h}{2\pi i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] \tag{208}$$

$$= -\frac{h^2}{4\pi^2}\left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] \tag{209}$$

$$= -\frac{h^2}{4\pi^2}\left(x\frac{\partial}{\partial y}z\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}x\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}z\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}x\frac{\partial}{\partial z}\right) \tag{210}$$

$$= -\frac{h^2}{4\pi^2}\left(xz\frac{\partial^2}{\partial yx} - x^2\frac{\partial^2}{\partial yz} - yz\frac{\partial^2}{\partial x^2} + yx\frac{\partial^2}{\partial xz} + y\frac{\partial}{\partial z}\frac{\partial x}{\partial x}\right) \tag{211}$$

$$= -\hbar^2\left(xz\frac{\partial^2}{\partial yx} - x^2\frac{\partial^2}{\partial yz} - yz\frac{\partial^2}{\partial x^2} + yx\frac{\partial^2}{\partial xz} + y\frac{\partial}{\partial z}\right) \tag{212}$$

Now putting the values of $\hat{L}_y\hat{L}_z$ and $\hat{L}_z\hat{L}_y$ in equation (212), we get the following.

$$[\hat{L}_y, \hat{L}_z] = \left[-\hbar^2\left(z\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial xy} - x^2\frac{\partial^2}{\partial^2 zy} - zy\frac{\partial}{\partial x^2} + xy\frac{\partial^2}{\partial zx}\right)\right]$$
$$- \left[-\hbar^2\left(xz\frac{\partial^2}{\partial yx} - x^2\frac{\partial^2}{\partial yz} - yz\frac{\partial^2}{\partial x^2} + yx\frac{\partial^2}{\partial xz} + y\frac{\partial}{\partial z}\right)\right] \quad (213)$$

$$[\hat{L}_y, \hat{L}_z] = -\hbar^2\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \quad (214)$$

Taking negative sign common, we get

$$[\hat{L}_y, \hat{L}_z] = \hbar^2\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad (215)$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \quad (216)$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \quad (217)$$

iii) The commutator of orbital angular momentum operators along *z* and *x*-axis.

$$[\hat{L}_z, \hat{L}_x] = \hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z \quad (218)$$

Finding the values of $\hat{L}_z\hat{L}_x$, we get

$$\hat{L}_z\hat{L}_x = \left[\frac{h}{2\pi i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\left[\frac{h}{2\pi i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \quad (219)$$

$$= -\frac{h^2}{4\pi^2}\left[\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \quad (220)$$

$$= -\frac{h^2}{4\pi^2}\left(x\frac{\partial}{\partial y}y\frac{\partial}{\partial z} - x\frac{\partial}{\partial y}z\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}y\frac{\partial}{\partial z} + y\frac{\partial}{\partial x}z\frac{\partial}{\partial y}\right) \quad (221)$$

$$= -\frac{h^2}{4\pi^2}\left(x\frac{\partial}{\partial z}\frac{\partial y}{\partial y} + xy\frac{\partial^2}{\partial yz} - xz\frac{\partial^2}{\partial y^2} - y^2\frac{\partial^2}{\partial xz} + yz\frac{\partial^2}{\partial xy}\right) \quad (222)$$

$$= -\hbar^2\left(x\frac{\partial}{\partial z} + xy\frac{\partial^2}{\partial yz} - xz\frac{\partial^2}{\partial y^2} - y^2\frac{\partial^2}{\partial xz} + yz\frac{\partial^2}{\partial xy}\right) \quad (223)$$

Similarly obtaining the value of $\hat{L}_x\hat{L}_z$, we get

$$\hat{L}_x\hat{L}_z = \left[\frac{h}{2\pi i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\left[\frac{h}{2\pi i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \quad (224)$$

$$= -\frac{h^2}{4\pi^2}\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \quad (225)$$

or

$$= -\frac{h^2}{4\pi^2}\left(y\frac{\partial}{\partial z}x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}x\frac{\partial}{\partial y} + z\frac{\partial}{\partial y}y\frac{\partial}{\partial x}\right) \quad (226)$$

$$= -\frac{h^2}{4\pi^2}\left(yx\frac{\partial^2}{\partial zy} - y^2\frac{\partial^2}{\partial zx} - zx\frac{\partial^2}{\partial y^2} + z\frac{\partial}{\partial x}\frac{\partial y}{\partial y} + zy\frac{\partial^2}{\partial yx}\right) \quad (227)$$

$$= -\hbar^2\left(yx\frac{\partial^2}{\partial zy} - y^2\frac{\partial^2}{\partial zx} - zx\frac{\partial^2}{\partial y^2} + z\frac{\partial}{\partial x} + zy\frac{\partial^2}{\partial yx}\right) \quad (228)$$

Now putting the values of $\hat{L}_z\hat{L}_x$ and $\hat{L}_x\hat{L}_z$ in equation (218), we get the following.

$$[\hat{L}_z, \hat{L}_x] = \left[-\hbar^2\left(x\frac{\partial}{\partial z} + xy\frac{\partial^2}{\partial yz} - xz\frac{\partial^2}{\partial y^2} - y^2\frac{\partial^2}{\partial xz} + yz\frac{\partial^2}{\partial xy}\right)\right] - \left[-\hbar^2\left(yx\frac{\partial^2}{\partial zy} - y^2\frac{\partial^2}{\partial zx} - zx\frac{\partial^2}{\partial y^2} + z\frac{\partial}{\partial x} + zy\frac{\partial^2}{\partial yx}\right)\right] \quad (229)$$

$$[\hat{L}_z, \hat{L}_x] = -\hbar^2\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) \quad (230)$$

Taking negative sign common, we get

$$[\hat{L}_z, \hat{L}_x] = \hbar^2\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad (231)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] \quad (232)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \quad (233)$$

iv) The commutator of total orbital angular momentum squared operator and orbital angular momentum along one of the three-axis.

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z] \quad (234)$$

$$= [\hat{L}_x^2\,\hat{L}_z + \hat{L}_y^2\,\hat{L}_z + \hat{L}_z^2\,\hat{L}_z - \hat{L}_z\hat{L}_x^2 - \hat{L}_z\hat{L}_y^2 - \hat{L}_z\hat{L}_z^2] \quad (235)$$

$$= [(\hat{L}_x^2\,\hat{L}_z - \hat{L}_z\hat{L}_x^2) + (\hat{L}_y^2\,\hat{L}_z - \hat{L}_z\hat{L}_y^2) + (\hat{L}_z^2\,\hat{L}_z - \hat{L}_z\hat{L}_z^2)] \quad (236)$$

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2\,\hat{L}_z] + [\hat{L}_z^2\,\hat{L}_z] \quad (237)$$

Now finding $[\hat{L}_x^2, \hat{L}_z]$ first, we get

$$[\hat{L}_x^2, \hat{L}_z] = \hat{L}_x^2 \hat{L}_z - \hat{L}_z\hat{L}_x^2 \tag{238}$$

$$= \hat{L}_x\hat{L}_x \hat{L}_z - \hat{L}_z\hat{L}_x\hat{L}_x \tag{239}$$

$$= [\hat{L}_x\hat{L}_x \hat{L}_z - \hat{L}_x\hat{L}_z \hat{L}_x] - [\hat{L}_z\hat{L}_x\hat{L}_x - \hat{L}_x\hat{L}_z \hat{L}_x] \tag{240}$$

$$= \hat{L}_x [\hat{L}_x\hat{L}_z - \hat{L}_z \hat{L}_x] - [\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z]\hat{L}_x \tag{241}$$

$$= \hat{L}_x [\hat{L}_x\hat{L}_z - \hat{L}_z \hat{L}_x] + [\hat{L}_x\hat{L}_z - \hat{L}_z\hat{L}_x]\hat{L}_x \tag{242}$$

$$= \hat{L}_x [-i\hbar\hat{L}_y] + [-i\hbar\hat{L}_y]\hat{L}_x \tag{243}$$

$$= -i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_x = -i\hbar[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x] \tag{244}$$

Similarly,

$$[\hat{L}_y^2, \hat{L}_z] = \hat{L}_y^2 \hat{L}_z - \hat{L}_z\hat{L}_y^2 \tag{245}$$

$$= \hat{L}_y\hat{L}_y \hat{L}_z - \hat{L}_z\hat{L}_y\hat{L}_y \tag{246}$$

$$= [\hat{L}_y\hat{L}_y \hat{L}_z - \hat{L}_y\hat{L}_z\hat{L}_y] - [\hat{L}_z\hat{L}_y\hat{L}_y - \hat{L}_y\hat{L}_z\hat{L}_y]$$

$$= \hat{L}_y [\hat{L}_y\hat{L}_z - \hat{L}_z \hat{L}_y] - [\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z]\hat{L}_y$$

$$= \hat{L}_y [\hat{L}_y\hat{L}_z - \hat{L}_z \hat{L}_y] + [\hat{L}_y\hat{L}_z - \hat{L}_z \hat{L}_y]\hat{L}_y$$

$$= \hat{L}_y [i\hbar\hat{L}_x] + [i\hbar\hat{L}_x]\hat{L}_y$$

$$= i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_x\hat{L}_y = i\hbar[\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y] \tag{247}$$

Similarly,

$$[\hat{L}_z^2, \hat{L}_z] = \hat{L}_z^2 \hat{L}_z - \hat{L}_z\hat{L}_z^2$$

$$= \hat{L}_z\hat{L}_z \hat{L}_z - \hat{L}_z\hat{L}_z\hat{L}_z$$

$$[\hat{L}_z^2, \hat{L}_z] = 0 \tag{248}$$

Now putting the value of $\hat{L}_x^2\hat{L}_z$, $\hat{L}_y^2\hat{L}_z$ and $\hat{L}_z^2\hat{L}_z$ in equation (237), we get

$$[\hat{L}^2, \hat{L}_z] = -i\hbar[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x] + i\hbar[\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y] + 0$$

$$[\hat{L}^2, \hat{L}_z] = 0 \tag{249}$$

Also

$$[\hat{L}^2, \hat{L}_y] = 0; and\ [\hat{L}^2, \hat{L}_x] = 0 \quad (250)$$

Hence, the commutation relations of angular momentum operators along two different directions do not commute with each other and hence cannot give eigenvalues simultaneously and accurately. One the other hand, total angular momentum squared and angular momentum along one axis do commute with each other.

The commutation relations between angular momentum operators can be mainly divided into four categories as discussed below.

(a) Orbital angular momentum commutation:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z; \quad [\hat{L}_y, \hat{L}_x] = -i\hbar\hat{L}_z \quad (251)$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x; \quad [\hat{L}_z, \hat{L}_y] = -i\hbar\hat{L}_x \quad (252)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y; \quad [\hat{L}_x, \hat{L}_z] = -i\hbar\hat{L}_y \quad (253)$$

$$[\hat{L}^2, \hat{L}_x] = 0; \quad [\hat{L}_x, \hat{L}^2] = 0 \quad (254)$$

$$[\hat{L}^2, \hat{L}_y] = 0; \quad [\hat{L}_y, \hat{L}^2] = 0 \quad (255)$$

$$[\hat{L}^2, \hat{L}_z] = 0; \quad [\hat{L}_z, \hat{L}^2] = 0 \quad (256)$$

(b) Spin angular momentum commutation:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z; \quad [\hat{S}_y, \hat{S}_x] = -i\hbar\hat{S}_z \quad (257)$$

$$[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x; \quad [\hat{S}_z, \hat{S}_y] = -i\hbar\hat{S}_x \quad (258)$$

$$[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y; \quad [\hat{S}_x, \hat{S}_z] = -i\hbar\hat{S}_y \quad (259)$$

$$[\hat{S}^2, \hat{S}_x] = 0; \quad [\hat{S}_x, \hat{S}^2] = 0 \quad (260)$$

$$[\hat{S}^2, S_y] = 0; \quad [\hat{S}_y, \hat{S}^2] = 0 \quad (261)$$

$$[\hat{S}^2, \hat{S}_z] = 0; \quad [\hat{S}_z, \hat{S}^2] = 0 \quad (262)$$

(c) Total angular momentum commutation:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z; \quad [\hat{J}_y, \hat{J}_x] = -i\hbar\hat{J}_z \quad (263)$$

$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x; \quad [\hat{J}_z, \hat{J}_y] = -i\hbar\hat{J}_x \quad (264)$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y; \quad [\hat{J}_x, \hat{J}_z] = -i\hbar\hat{J}_y \quad (265)$$

$$[\hat{J}^2, \hat{J}_x] = 0; \quad [\hat{J}_x, \hat{J}^2] = 0 \quad (266)$$

$$[\hat{J}^2, J_y] = 0; \quad [\hat{J}_y, \hat{J}^2] = 0 \tag{267}$$

$$[\hat{J}^2, \hat{J}_z] = 0; \quad [\hat{J}_z, \hat{J}^2] = 0 \tag{268}$$

(d) Total angular momentum commutation:

$$[\hat{L}_x, \hat{S}_x] = 0; \quad [\hat{S}_x, \hat{L}_x] = 0 \tag{263}$$

$$[\hat{L}_x, \hat{S}_y] = 0; \quad [\hat{S}_y, \hat{L}_x] = 0 \tag{264}$$

$$[\hat{L}_x, \hat{S}_z] = 0; \quad [\hat{S}_z, \hat{L}_x] = 0 \tag{265}$$

$$[\hat{L}_y, \hat{S}_x] = 0; \quad [\hat{S}_x, \hat{L}_y] = 0 \tag{266}$$

$$[\hat{L}_y, \hat{S}_y] = 0; \quad [\hat{S}_y, \hat{L}_y] = 0 \tag{267}$$

$$[\hat{L}_y, \hat{S}_z] = 0; \quad [\hat{S}_z, \hat{L}_y] = 0 \tag{268}$$

$$[\hat{L}_z, \hat{S}_x] = 0; \quad [\hat{S}_x, \hat{L}_z] = 0 \tag{269}$$

$$[\hat{L}_z, \hat{S}_y] = 0; \quad [\hat{S}_y, \hat{L}_z] = 0 \tag{270}$$

$$[\hat{L}_z, \hat{S}_z] = 0; \quad [\hat{S}_z, \hat{L}_z] = 0 \tag{271}$$

4. Commutators of Ladder operators:

i) Find the commutator of the following

$$[\hat{J}^2, \hat{J}_+] \tag{272}$$

Let

$$[\hat{J}^2, \hat{J}_+] = [\hat{J}^2, \hat{J}_x + i\hat{J}_y] \tag{273}$$

$$= \hat{J}^2(\hat{J}_x + i\hat{J}_y) - (\hat{J}_x + i\hat{J}_y)\hat{J}^2 \tag{274}$$

$$= \hat{J}^2\hat{J}_x + i\hat{J}^2\hat{J}_y - \hat{J}_x\hat{J}^2 - i\hat{J}_y\hat{J}^2 \tag{275}$$

$$= [\hat{J}^2\hat{J}_x - \hat{J}_x\hat{J}^2] + i[\hat{J}^2\hat{J}_y - \hat{J}_y\hat{J}^2] \tag{276}$$

$$= [\hat{J}^2, \hat{J}_x] + i[\hat{J}^2, \hat{J}_y] \tag{277}$$

$$= 0 + i(0) = 0 \tag{278}$$

Hence

$$[\hat{J}^2, \hat{J}_+] = 0 \tag{279}$$

Similarly

$$[\hat{J}^2, \hat{J}_-] = 0 \tag{280}$$

ii) Find the commutator of the following

$$[\hat{J}_+, \hat{J}_z] \tag{281}$$

Let

$$[\hat{J}_+, \hat{J}_z] = [\hat{J}_x + i\hat{J}_y, \hat{J}_z] \tag{282}$$

$$= (\hat{J}_x + i\hat{J}_y)\hat{J}_z - \hat{J}_z(\hat{J}_x + i\hat{J}_y) \tag{283}$$

$$= \hat{J}_x\hat{J}_z + i\hat{J}_y\hat{J}_z - \hat{J}_z\hat{J}_x - \hat{J}_z i\hat{J}_y \tag{284}$$

$$= \hat{J}_x\hat{J}_z - \hat{J}_z\hat{J}_x + i\hat{J}_y\hat{J}_z - i\hat{J}_z\hat{J}_y \tag{285}$$

$$= [\hat{J}_x\hat{J}_z - \hat{J}_z\hat{J}_x] + i[\hat{J}_y\hat{J}_z - \hat{J}_z\hat{J}_y] \tag{286}$$

$$= [\hat{J}_x, \hat{J}_z] + i[\hat{J}_y, \hat{J}_z] \tag{287}$$

$$= -i\hbar\hat{J}_y + i(i\hbar\hat{J}_x) = -i\hbar\hat{J}_y - \hbar\hat{J}_x \tag{288}$$

$$= -\hbar(\hat{J}_x + i\hat{J}_y) = -\hbar\hat{J}_+ \tag{289}$$

$$[\hat{J}_+, \hat{J}_z] = -\hbar\hat{J}_+ \tag{290}$$

Similarly

$$[\hat{J}_-, \hat{J}_z] = \hbar\hat{J}_- \tag{291}$$

iii) Find the commutator of the following

$$[\hat{J}_+, \hat{J}_-] \tag{292}$$

Let

$$[\hat{J}_+, \hat{J}_-] = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) - (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) \tag{293}$$

$$= \hat{J}_x\hat{J}_x - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x + \hat{J}_y\hat{J}_y - (\hat{J}_x\hat{J}_x + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x + \hat{J}_y\hat{J}_y) \tag{294}$$

$$= \hat{J}_x\hat{J}_x - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x + \hat{J}_y\hat{J}_y - \hat{J}_x\hat{J}_x - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x - \hat{J}_y\hat{J}_y \tag{295}$$

$$= -i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x \tag{296}$$

$$= -i[\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x] + i[\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y] \tag{297}$$

$$= -i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x] \tag{298}$$

$$= -i[i\hbar\hat{J}_z] + i[-i\hbar\hat{J}_z] \tag{299}$$

$$= \hbar\hat{J}_z + \hbar\hat{J}_z = 2\hbar\hat{J}_z \tag{300}$$

The commutation relations between angular-momentum and Ladder operators can be mainly divided into three categories as discussed below.

(a) Ladder operator and total angular momentum commutation:

$$[\hat{J}^2, \hat{J}_+] = 0; \qquad [\hat{J}_+, \hat{J}^2] = 0 \tag{301}$$

$$[\hat{J}^2, \hat{J}_-] = 0; \qquad [\hat{J}_-, \hat{J}^2] = 0 \tag{302}$$

$$[\hat{J}_+, \hat{J}_z] = -\hbar\hat{J}_+; \qquad [\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+ \tag{303}$$

$$[\hat{J}_-, \hat{J}_z] = \hbar\hat{J}_-; \qquad [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_- \tag{304}$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z; \qquad [\hat{J}_-, \hat{J}_+] = -2\hbar\hat{J}_z \tag{305}$$

(b) Ladder operator and orbital angular momentum commutation:

$$[\hat{L}^2, \hat{L}_+] = 0; \qquad [\hat{L}_+, \hat{L}^2] = 0 \tag{306}$$

$$[\hat{L}^2, \hat{L}_-] = 0; \qquad [\hat{L}_-, \hat{L}^2] = 0 \tag{307}$$

$$[\hat{L}_+, \hat{L}_z] = -\hbar\hat{L}_+; \qquad [\hat{L}_z, \hat{L}_+] = \hbar\hat{L}_+ \tag{308}$$

$$[\hat{L}_-, \hat{L}_z] = \hbar\hat{L}_-; \qquad [\hat{L}_z, \hat{L}_-] = -\hbar\hat{L}_- \tag{309}$$

$$[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z; \qquad [\hat{L}_-, \hat{L}_+] = -2\hbar\hat{L}_z \tag{310}$$

(b) Ladder operator and spin angular momentum commutation:

$$[\hat{S}^2, \hat{S}_+] = 0; \qquad [\hat{S}_+, \hat{S}^2] = 0 \tag{311}$$

$$[\hat{S}^2, \hat{S}_-] = 0; \qquad [\hat{S}_-, \hat{S}^2] = 0 \tag{312}$$

$$[\hat{S}_+, \hat{S}_z] = -\hbar\hat{S}_+; \qquad [\hat{S}_z, \hat{S}_+] = \hbar\hat{S}_+ \tag{313}$$

$$[\hat{S}_-, \hat{S}_z] = \hbar\hat{S}_-; \qquad [\hat{S}_z, \hat{S}_-] = -\hbar\hat{S}_- \tag{314}$$

$$[\hat{S}_+, \hat{S}_-] = 2\hbar\hat{S}_z; \qquad [\hat{S}_-, \hat{S}_+] = -2\hbar\hat{S}_z \tag{315}$$

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ISBN: 978-81-938720-1-7
9 788193 872017 >
MRP: Rs 800.00

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